NUMERICAL IDENTIFICATION OF ELASTICITY COEFFICIENTS FOR THE BENDING PROBLEM

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Abstract

A test for identifying elasticity coefficients for the bending problem is discussed. Bending is a problem of flat textiles, and its existence is demonstrated by experimental material on the basis of coordinates of a bent sample. In this paper, we propose another usage of Peirce's cantilever test for determining the relationship between bending moment and curvature. Measuring error was taken into consideration during analysis. The results of calculation have been limited to the relationship between bending moment and curvature mainly of the second-degree, but in this way the relationships of a higher degree can be examined. For the analysis a special computer program was developed, which can calculate the coordinates of a bent sample on the basis of an image saved as a .bmp file (bitmap).

Keywords

Bending rigidity, elasticity coefficients, fabrics, textile mechanics, bending test, numerical methods, numerical identification, mechanical properties, woven fabric, image analysis

1. Introduction

From the aesthetic viewpoint, the bending rigidity of textiles is an essential property. Fibres are elastoplastic bodies. In woven fabrics, fibres occur in combined form, firstly in yarn, then as a result of weaving. Deflections of fabrics are comparatively large, and therefore the limit of proportionality is exceeded in many cases. For this reason, establishing the close relationships between the geometry of deflection and mechanical quantities is very difficult. It requires complicated theoretical considerations supported by experimental material. Analyses of bending rigidity always refer to experimental results. Research methods can be divided into two groups: gravitational methods and rotational methods. The common feature of these methods is that experimental relationships between the geometry of deflection and the bending moment are always obtained. These methods call for much measurement of the different values of bending moment, as well as appropriate data handling.

This work deals with numerical identification of the bending problem for textiles. The identification concerns the physical relationship between bending moment $M$ and curvature $1/\rho$. Many experiments show that this relationship is not generally linear, and can be presented in a more complex form. It will be assumed that the physics experiment concerns the cantilever test proposed by Peirce (Figure 1) and described in works [1], [2] and [3].

![Figure 1. The cantilever test proposed by Peirce](http://www.autexrj.org/No4-2004/0109.pdf)
On the basis of Peirce’s test, the researcher can obtain a certain measure of bending rigidity. This measure is the so-called bending length. In this paper another use of Peirce’s test for determining the relationship between bending moment and curvature is proposed. For that purpose, it is sufficient to carry out only one experiment. As a result of this experiment, the shape of the bent sample is obtained.

We obtain the values of coordinates \((x, y)\) of a bent sample of length \(l\) for following values of curvilinear coordinate \(s\). The coordinate \(s\) is measured along an axis starting from point A. Of course, several experiments may be carried out in order to verify the results.

For the analysis described, a special computer program was developed. This program can calculate the coordinates \((x, y)\) of a bent sample on the basis of a picture saved as a .bmp file (bitmap).

2. Numerical solution to the problem

Before beginning the analysis, the appropriate mathematical method should be chosen. It should also be noted that the obtained coordinates \((x, y)\) are burdened with errors. These errors arise from the testing method and also from the calculation of the coordinates. The coordinates \((x, y)\) can be obtained on the basis of image analysis. The image can be obtained as a photo of a bent sample.

In this work, integration of bending moment function is applied. This results from the fact that the differentiation of a function can be burdened with a large error in many cases.

2.1. Identification of relationship between bending moment and curvature

The analysis of the relationship between the curvature of a bent sample \(\frac{1}{\rho} = \frac{d\phi}{ds}\) and bending moment \(M\) will be described below. Many experiments show that this relationship is not generally linear, and can be presented in a more complex form. This relationship can be written as

\[
\frac{d\phi}{ds} = a_1 M + a_2 M^2 + a_3 M^3 + ... \tag{1}
\]

where \(M\) – the bending moment, \(\phi\) - the inclination angle of tangent to the axis of a bent sample.

The unknown temporary coefficients \(a_1, a_2, a_3, ...\) in Equation (1) can be called elasticity coefficients for the bending problem. In this work, the relationship (1) is limited to the/a second-degree polynomial. Thus

\[
\frac{d\phi}{ds} = a_1 M + a_2 M^2. \tag{2}
\]

Hence

\[
\phi = \int \left( a_1 M + a_2 M^2 \right) ds. \tag{3}
\]

The inclination angle \(\phi\) of the tangent to the axis of the sample at any point described by coordinate \(s\) is a function of three variables \(\phi(s, a_1, a_2)\).

From Figure 2, we obtain

\[
dx = \cos(\phi) ds, \quad dy = \sin(\phi) ds.
\]

Therefore the coordinates \(x\) and \(y\) can be obtained by integration.

\[
x = \int \cos(\phi) ds, \quad y = \int \sin(\phi) ds. \tag{4}
\]

Similar to the angle \(\phi\) coordinates \(x\) and \(y\) are the functions of three variables.

\[
x(s, a_1, a_2), \quad y(s, a_1, a_2). \tag{5}
\]
Thus, if we find a relationship for the bending moment $M$ on the basis of coordinates $x$ and $y$ we can calculate the theoretical values of the coordinates $x$ and $y$ from Equations (3) and (4) (these coordinates are the analytical solution).

![Figure 2. Infinitesimal segment $ds$](image)

For the identification, it is important that a difference between the experimental coordinates $x$ and $y$, represented as $(\bar{x}, \bar{y})$, and the analytical coordinates $(x, y)$, is the smallest possible for the whole length of sample $l$.

It depends, of course, on an appropriate choice of elasticity coefficients $a_1, a_2$ in Equation (2). The measure of matching can in this case be the function as below:

$$H = \int_0^l \left[ (x - \bar{x})^2 + (y - \bar{y})^2 \right] ds . \quad (6)$$

We look for the parameters $a_1, a_2$, for which function (6) reaches the minimum value ($H = \min$). After establishing these parameters, it is possible to claim that relationship (2) describes the bending of the sample tested.

### 2.2. Calculation of bending moment

At the stage of identifying the bending problem, accurate calculation of the bending moment $M$ plays an important role. This moment should be calculated on the basis of the coordinates $(x, y)$, that is, on the basis of geometry of a bent sample. It should also be noted that we must still have the weight per unit length of sample $q$, which is from the constant assumption for the whole length of sample $l$. The coordinates $(x, y)$ are in this case the functions of curvilinear coordinate $s$, which is measured along the axis starting from point A ($s_A = 0$).

![Figure 3. The sample split into $n$ elements](image)

In Figure 3, the scheme of partition of the sample is presented. We have $n$ points starting from A. For Point A an index '0' is assumed. After geometrical analysis of the shape, we have the values of coordinates $x_i(s), y_i(s)$ for $i = 0,1,2,\ldots,n$. The bending moment at any point is the algebraic sum of all moments of forces of gravity for each element $ds$ on one side of the point under consideration.
(starting from point B to $i$). The individual force of gravity $qds$ acts at the centre of segment $ds$. Therefore the formula for bending moment at any point $i$ is given by

$$M_i = qds \left[ \sum_{j=i+1}^{n} \frac{x_j + x_{j-1}}{2} \right] - (n-i)x_i. \quad (7)$$

Equation (7) is derived on the assumption that the length of segment $ds$ is constant ($ds = \text{const}$.). In most cases, the partition of the sample is non-uniform. The length of segment $ds$ depends on the coordinates of its ends. In this case, the bending moment at point $i$ is given by

$$M_i = q \sum_{j=i+1}^{n} \left[ \frac{x_j + x_{j-1}}{2} - x_i \right] \sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2}. \quad (8)$$

3. Numerical examples

In order to verify our consideration, two numerical examples are presented below. In these two cases we know the accurate relationship between bending moment and curvature.

**Example 1**

We have a sample of textile for which the relationship between bending moment and curvature is known. This relationship is exactly linear $\frac{d\phi}{ds} = \frac{M}{C}$. For this sample we have the following data:

- length of sample $l = 0.1$ m,
- constant bending rigidity $C = 10^{-5}$ Nm$^{-1}$,
- weight per unit length $q = 0.1$ N/m.

The exact relationship between bending moment and curvature after numerical data using is $\frac{d\phi}{ds} = aM$, where $a = 100,000 \text{ (Nm}^2\text{)}^{-1}$.

Numerical solution of the problem was carried out using the Mathematica 4.0 package by Wolfram Research (www.wolfram.com). Minimization of the objective function $H (6)$ was carried out using the standard minimisation procedure of the Mathematica package [4]. In this case coefficient $a$ was sought. This coefficient occurs in the following relationship $\frac{d\phi}{ds} = a_i M$. Additionally, the coordinates $(x, y)$ were burdened with a maximum error of 5%.

After the numerical solution we obtained the following results (units are omitted):

$$a_i = 99,501 \text{ (exact value } a = 100,000\text{), value of objective function } H = 4.07336 \cdot 10^{-5}.$$ 

The error of estimation is not large, about $\varepsilon = 0.5\%$.

**Example 2**

We have considered the sample for which the relationship between bending moment and curvature is second-degree $\frac{d\phi}{ds} = aM + bM^2$, where $a = 10^5 \text{ (Nm}^2\text{)}^{-1}$, $b = 2 \cdot 10^8 \text{ (N}^2\text{m}^3\text{)}^{-1}$. The other data is as above.

The coordinates $(x, y)$ were burdened with a maximum error of 5%. In this case coefficients $a_1, a_2$ were sought. These occur in the following relationship: $\frac{d\phi}{ds} = a_1 M + a_2 M^2$.

After the numerical solution we obtained the following results (units are omitted):

$$a_1 = 99,956, \ a_2 = 1.99 \cdot 10^8, \text{ value of objective function } H = 3.15646 \cdot 10^{-11}.$$ 

The error of estimation is not large.
4. Conclusion

The identification of elasticity coefficients presented in this work turned out to be effective. This analysis was carried out on the basis of the values of coordinates \((x, y)\) of a bent sample and related to the relationship between bending moment and curvature. During analysis, it was taken into consideration that the values of coordinates could be burdened with errors. The Mathematica 4.0 package was used for calculation, and it turned out to be a very fast and effective tool.

Using integration during analysis give results of satisfactory accuracy. This work has been limited to examining the relationships of second degree at most (2). Of course, we can consider the relationships of a higher degree than second in this same way.

References